Methods for determination of microwave cavity quality factors from equivalent electronic circuit models

Kevin D. McKinstry and Carl E. Patton

Department of Physics, Colorado State University, Fort Collins, Colorado 80523

(Received 7 September 1988; accepted for publication 14 November 1988)

Simple analytic methods have been developed for high-accuracy measurement of the quality factor Q of reflection and transmission microwave cavities. These methods are based on the consideration of cavity equivalent electronic circuit models and use measured cavity parameters over a narrow frequency range spanning the cavity resonance frequency to determine the Q factor. Since only straightforward frequency and power measurements are necessary, these methods are well suited to computer-aided data acquisition and analysis. Specific results are presented for a standard X-band reflection cavity with a Q of 2699.1 ± 0.5 and a high-Q X-band cylindrical cavity with a Q of 23112 ± 4.

INTRODUCTION

Textbook procedures for determining the quality factor Q of microwave cavities are often time consuming and of limited accuracy. These procedures for reflection cavities usually involve the measurement of a voltage standing wave ratio (VSWR) or reflection coefficient versus frequency over a range of frequencies which straddle the cavity resonance frequency.[1] Graphical data analyses or iterative procedures are then used to determine the frequency half-power points, and hence the cavity Q. The usual method for determining the Q of transmission cavities involves measurement of frequency points off the main cavity resonance where the transmitted power is at half-maximum.[2] While simple in concept, this method is sometimes unreliable due to errors in power and frequency measurement. Frequency resolution and temperature drift of the cavity response can also affect the accuracy of Q measurements, especially for high Q cavities.

The methods described in this article to find the cavity Q are an extension of previous work by Frait and Patton on reflection cavities.[3] Several typographical errors in Ref. 3 have been corrected. These methods involve neither of the abovementioned procedures. Instead, transmission or reflection coefficient versus frequency data, for transmission or reflection cavities, respectively, are obtained over a small frequency range straddling the cavity resonance frequency ω_r. The Q parameters in each case are then found by a linear least-square fit of reduced equivalent circuit parameters obtained from the data. These techniques are similar to the "straight-line" method discussed by Sucher.[4]

I. REFLECTION CAVITY

The method of determining the cavity Q for a reflection cavity is based on (1) an equivalent circuit analysis and (2) the observation that the cavity reflection coefficient ρ may be expressed in terms of the line impedance Z_l and the cavity impedance Z_c, which in turn may be expressed in terms of the external, volume, and loaded Q factors, Q_E, Q_V, and Q_L, respectively. The relevant Q factors may be obtained therefore by an analysis of the reflection cavity circuit.

The reflection coefficient ρ is defined by

$$\rho = V_{OR}/V_{OI}, \tag{1}$$

where V_{OI} is the voltage amplitude for the incident signal in the transmission line feeding the cavity and V_{OR} is the amplitude of the reflected voltage. In terms of the line impedance Z_l and the cavity impedance Z_c the reflection coefficient may be written as[2]

$$\rho = \frac{|Z_c - Z_l|}{|Z_c + Z_l|}. \tag{2}$$

Expressions for these impedances in terms of the cavity Q factors can be obtained from an equivalent circuit analysis. Consider the circuit model for a reflection cavity with an inductive iris as shown in Fig. 1(a). In this figure, V is the voltage amplitude for the input drive signal at frequency ω (rad/s), Z_l is the line impedance, L_i is the iris inductance, n is the transformer equivalent iris coupling coefficient, and R, L, and C are the equivalent circuit cavity resistance, inductance, and capacitance, respectively. The transfer circuit for the cavity is shown in Fig. 1(b). The total circuit impedance of the cavity, based on the transfer circuit, is given by

$$Z = (n^2 Z_l) + R + j[\omega(L + n^2 L_i) - 1/\omega C], \tag{3}$$

and the cavity resonance frequency (in rad/s) is given by

$$\omega_r = [1/(L + n^2 L_i)C]^{1/2}. \tag{4}$$

In terms of this ω_r, Z may be written as

$$Z = (n^2 Z_l) + R + j\omega_r(L + n^2 L_i)(\omega/\omega_r - \omega_r/\omega). \tag{5}$$

It will be useful to identify and define a reduced frequency parameter Δ, given by

$$\Delta = \omega/\omega_r - \omega_r/\omega. \tag{6}$$

We note that ω_r is a measurable quantity.

Following the usual convention,[1,4] the power dissipation associated with the $n^2 Z_l$ element in the equivalent circuit is characterized by an external Q factor Q_E, and the dissipation associated with the R element is characterized by a volume Q factor Q_V. For the actual cavity, Q_E and Q_V are associated with the external iris losses and the losses inside the cavity volume, respectively. In terms of the circuit pa-

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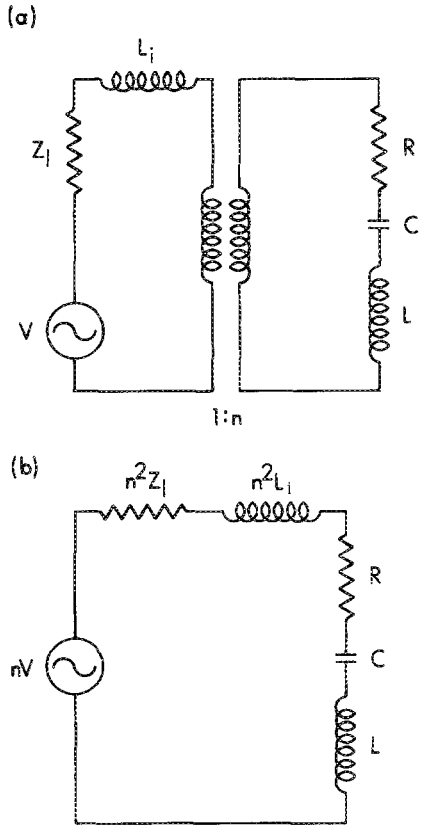

FIG. 1. (a) Equivalent circuit for an inductively coupled reflection cavity. (b) Transfer circuit for the circuit in (a). In both parts, V is the drive amplitude, Z_l is the line impedance, L_a is the iris aperture inductance, n is the iris coupling coefficient, and R, L, and C are the equivalent circuit resistance, inductance, and capacitance, respectively.

rameters and the resonance frequency ω_r, these Q factors are given by

$$Q_E = \omega_r(L + n^2L_i)/n^2Z_l, \tag{7}$$

$$Q_V = \omega_r(L + n^2L_i)/R. \tag{8}$$

The overall Q factor Q_L, often termed the loaded Q, can be obtained from the relation

$$\frac{1}{Q_L} = \frac{1}{Q_E} + \frac{1}{Q_V}. \tag{9}$$

The above equations provide the basis for obtaining three operationally useful reduced parameters, δ_E, δ_V, and δ_L, which satisfy the equations

$$\delta_E = Q_E\Delta, \tag{10}$$

$$\delta_V = Q_V\Delta, \tag{11}$$

and

$$\delta_L = Q_L\Delta. \tag{12}$$

Expressions for δ_E, δ_V, and δ_L in terms of the measurable quantities ω and ρ will be derived below. Given ρ vs ω data and the corresponding reduced data on δ_E, δ_V, and δ_L vs Δ, linear least-square fits can be used to determine the cavity Q factors in a very straightforward manner.

In order to obtain expressions for the δ parameters in terms of ρ and ω, we first write the cavity reflection coefficient in terms of the effective cavity impedance and the effective line impedance. From Eq. (5), we have

$$Z_c^{\text{eff}} = R + jRQ_V\Delta \tag{13}$$

and

$$Z_l^{\text{eff}} = n^2Z_l. \tag{14}$$

The voltage reflection coefficient ρ in Eq. (2) may now be written in terms of these effective impedances as

$$\rho = \frac{|1 + jQ_V\Delta - n^2Z_l/R|}{|1 + jQ_V\Delta + n^2Z_l/R|}. \tag{15}$$

We now take advantage of the conventional definition of the cavity coupling coefficient β,

$$\beta = \frac{Q_V}{Q_E} = \left(\frac{1 \mp \rho_r}{1 \pm \rho_r}\right), \tag{16}$$

where the ρ_r parameter corresponds to the value of the cavity reflection coefficient ρ at the cavity resonance frequency, $\omega = \omega_r$. The upper and lower signs in Eq. (16) apply to undercoupled ($\beta < 1$) and overcoupled ($\beta > 1$) cavities, respectively. At critical coupling $\beta = 1$, $\rho_r = 0$, and the relation $Q_E = Q_V$ is satisfied. We note that ρ_r is a measurable quantity (the value of ρ at $\omega = \omega_r$) and, hence, β may be determined empirically.

We may now eliminate the unknown circuit parameters R and n^2Z in Eq. (15) by invoking the definition of β in terms of Q_E and Q_V, along with the circuit equivalent expressions for Q_E and Q_V in Eqs. (7) and (8). The coupling coefficient becomes

$$\beta = n^2Z_l/R. \tag{17}$$

With β from Eq. (17) and the δ_V definition of Eq. (11), we may now write

$$\rho = \sqrt{\frac{(1-\beta)^2 + \delta_V^2}{(1+\beta)^2 + \delta_V^2}} \tag{18}$$

and

$$\delta_V = \pm\sqrt{\frac{(1+\beta)^2\rho^2 - (1-\beta)^2}{1-\rho^2}}. \tag{19}$$

The $\pm$ sign is determined by the sign of Δ, which is negative for $\omega < \omega_r$ and positive for $\omega > \omega_r$. The above analysis yields the quantities δ_V and Δ in terms of the measured quantities ω, ρ, and ρ_r, and the nature of the cavity, undercoupled or overcoupled. One may therefore use data on ρ vs ω to obtain reduced data in the form of δ_V vs Δ, and determine Q_V from the slope of a straight-line fit to those reduced data as described above.

The remaining δ parameters δ_E and δ_L are related to δ_V by the equations

$$\delta_E = \delta_V/\beta \tag{20}$$

and

$$\delta_L = \delta_V/(1+\beta), \tag{21}$$

with δ_V as given by Eq. (19). Linear fits to reduced data on δ_E and δ_L vs Δ yield determinations of Q_E and Q_L in the same manner as described above.

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The above analysis yields a straightforward step-by-step procedure for the accurate determination of the Q factor for a reflection cavity:

(1) Determination whether the cavity is undercoupled or overcoupled. This procedure is discussed in detail in Ref. 5. In practice, however, if the cavity is close to critical coupling ($\rho_r = 0$) the loaded Q, Q_L, will be the same whether the cavity is considered undercoupled or overcoupled.

(2) Measure the power-reflection coefficient ρ over a frequency interval which straddles the cavity resonance frequency ω_r.

(3) Determine accurate value for the resonance frequency ω_r by finding the zero point of the curve of $d\rho/d\omega$ vs ω. Since the ρ vs ω curve is smoothly varying and has one maximum (at ω_r), the frequency where the derivative goes to zero is ω_r. The parameter ρ_r is then found from the $\rho(\omega)$ data at ω_r.

(4) Calculate the coupling coefficient $\beta = (1-\rho_r)/(1+\rho_r)$ (undercoupled) or $\beta = (1+\rho_r)/(1-\rho_r)$ (overcoupled).

(5) Calculate $\Delta(\omega,\omega_r)$ and then determine δ_V, δ_E, and δ_L vs Δ for the ensemble of ρ vs ω data.

(6) Do a linear least-square fit of δ_V, δ_E, and δ_L vs Δ. The slopes correspond to Q_V, Q_E, and Q_L.

As an example, the above procedure was followed to obtain Q factors for an x-band, undercoupled TE102 reflection cavity with a nominal resonance frequency of 9.4 GHz. A synthesized microwave sweeper was used to scan over a frequency interval of 2 MHz spanning the resonance frequency ω_r. The reflected power versus frequency was monitored with a scalar network analyzer. The measured frequency was calibrated using a microwave counter accurate to 10 Hz. A computer was used to control the sweeper, the analyzer, and the counter, and to manipulate the collected data. Over a 2-s sweep time, 801 data points were obtained. Representative numerical data for the measured cavity frequency f (in GHz), and the reduced cavity parameters ρ, Δ, δ_V, δ_E, and δ_L, are given in Table I for every 60th data point.

TABLE I. Representative data for a TE102 rectangular reflection x-band cavity. Every 60th data point has been shown here. The nominal cavity frequency is 9.389 GHz. The columns are for frequency f, the cavity reflection coefficient ρ, the reduced frequency parameter Δ, and the reduced cavity parameters δ_V, δ_E, and δ_L.

f (GHz)	ρ	$\Delta(\times 10^{-4})$	δ_V	δ_E	δ_L
9.388 232	0.499	−2.10	−1.105	−1.195	−0.574
9.388 382	0.445	−1.80	−0.954	−1.033	−0.496
9.388 532	0.380	−1.50	−0.788	−0.852	−0.409
9.388 681	0.310	−1.20	−0.623	−0.674	−0.323
9.388 831	0.234	−0.80	−0.457	−0.495	−0.237
9.388 981	0.156	−0.50	−0.294	−0.318	−0.152
9.389 132	0.077	−0.20	−0.128	−0.138	−0.066
9.389 281	0.044	0.06	0.039	0.042	0.020
9.389 431	0.113	0.38	0.206	0.223	0.107
9.389 581	0.193	0.70	0.371	0.401	0.192
9.389 730	0.271	1.10	0.536	0.580	0.278
9.389 880	0.344	1.33	0.701	0.759	0.364
9.390 030	0.413	1.65	0.868	0.940	0.451
9.390 181	0.475	1.97	1.037	1.122	0.538

Figure 2 shows plots of δ_E, δ_V, and δ_L vs Δ for the cavity. It is important to note that the lines on this plot represent the 801 *actual data points, not fitted curves.* It is clear that our procedure yields reduced δ vs Δ data which are exceedingly linear. The cavity Q factors, Q_E, Q_V, and Q_L, correspond to the slopes of the lines, and the indicated Q errors are the calculated standard deviations from the linear fits. The very small errors are one more indication of the almost perfect linear relation between the δ parameters and Δ. In the example, $Q_L = 2699$ and $f_r = 9.389$ GHz yield a cavity frequency width at half-maximum of 3.48 MHz. This frequency width for the cavity is almost double the width of the frequency window used for the data accumulation and analysis. The above result demonstrates that extremely accurate Q factors can be obtained from a narrow frequency window without going out to the relatively distant half-power frequency points.

II. TRANSMISSION CAVITY

The transmission cavity Q factor can be inferred from transmitted power versus frequency measurements using a technique similar to that of the reflection cavity case. Since the transmitted power from the cavity as a function of frequency follows a Lorentzian curve, it is a straightforward process to determine the Q factor from such measurements. Such a Lorentzian line shape can be determined from consideration of the transmission cavity equivalent circuit.

Our analysis of the transmission cavity follows the treatment by Kurokawa,[6] based on the equivalent circuit model in Fig. 3(a). In the figure, V is the drive amplitude, Z_i and Z_0 are the input and output line impedances, L_i and L_0 are the input and output iris inductances, n_i and n_0 are the respective input and output iris coupling coefficients, and R,

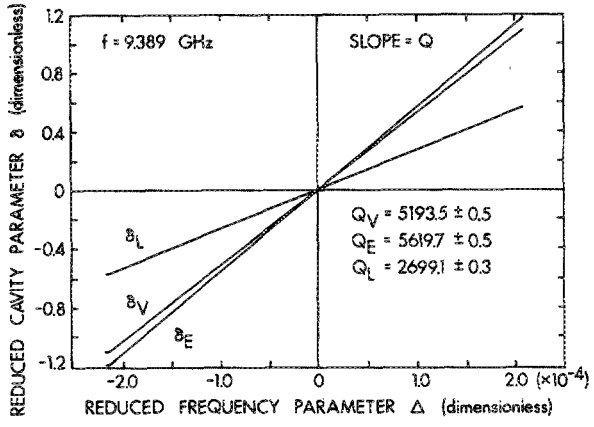

FIG. 2. Plot of reduced cavity parameters δ_L, δ_E, and δ_V vs the reduced frequency parameter Δ for the test reflection cavity. Note that the lines on this plot represent actual parameter points, not the fitted curves. The cavity Q parameters, Q_E, Q_V, and Q_L, correspond to the slopes of the lines, and the errors indicated are the calculated standard deviations from the least-square fits.

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L, and C are the equivalent circuit cavity resistance, inductance, and capacitance. The transfer circuit is shown in Fig. 3(b).

Transmitted power through the transfer circuit can be defined as that power dissipated through the output line impedance,

$$P(\omega) = n_0^2 Z_0 |I|^2, \tag{22}$$

where I is the current through the transfer circuit. In terms of the transfer circuit parameters, this power can be written as

$$P(\omega) = \frac{(n_i n_0 V)^2 Z_0}{\{n_i^2 Z_i + n_0^2 Z_0 + R + j[\omega(L + n_i^2 L_i + n_0^2 L_0) - 1/\omega C]\}^2}. \tag{23}$$

This expression can be simplified by use of the relationships between the transfer circuit parameters and the transmission cavity Q factors. These Q factors are associated with power dissipation through various elements of the transfer circuit. The dissipation associated with the $n_i^2 Z_i$ and $n_0^2 Z_0$ elements are characterized by external Q factors Q_{Ei} and Q_{E0}, respectively. The dissipation associated with the R element is characterized by a volume factor Q_V. These Q factors are given by

$$Q_{Ei} = \omega_r L_T / n_i^2 Z_i, \tag{24}$$

$$Q_{E0} = \omega_r L_T / n_0^2 Z_0, \tag{25}$$

and

$$Q_V = \omega_r L_T / R, \tag{26}$$

where the cavity resonance frequency ω_r is defined as

$$\omega_r = [(L + n_i^2 L_i + n_0^2 L_0) C]^{-1/2}, \tag{27}$$

and the total inductance L_T is

$$L_T = L + n_i^2 L_i + n_0^2 L_0. \tag{28}$$

The loaded Q factor Q_L, can then be obtained from the relation

$$\frac{1}{Q_L} = \frac{1}{Q_{Ei}} + \frac{1}{Q_{E0}} + \frac{1}{Q_V}. \tag{29}$$

Using Eqs. (20)–(25), the expression for the transmitted power can be written in Lorentzian form as

$$P(\omega) = P_r/(1 + Q_L^2 \Delta^2), \tag{30}$$

where the transmitted power at resonance P_r is

$$P_r = Z_0 (n_i n_0 V Q_L / \omega_r L_T)^2, \tag{31}$$

and the parameter Δ is as described in Eq. (13). Equation (30) may be expressed in an operationally useful form as

$$\tau = Q_L \Delta, \tag{32}$$

where the reduced transmission parameter τ is defined as

$$\tau = \pm \sqrt{P_r/P(\omega) - 1}. \tag{33}$$

As for the reflection cavity case, the $\pm$ sign is determined by Δ, which is negative for $\omega < \omega_r$ and positive for $\omega > \omega_r$.

The above analysis yields the quantities τ and Δ in terms of the measured quantities ω, p, and the P value at resonance, P_r. One may therefore plot the reduced data τ vs Δ and determine Q_L from the slope of a straight-line fit to those data. A straightforward procedure can therefore be used to determine an accurate value for the loaded cavity Q, Q_L. The procedure is similar to that described earlier for reflection cavities.

(1) Measure the transmitted power $P(\omega)$ as a function of frequency over a frequency range spanning the cavity resonance frequency ω_r.

(2) An accurate value for the resonance frequency ω_r is found by finding the zero point of the curve of $dP/d\omega$ vs ω. Since the P vs ω curve is smoothly varying and has one maximum (at ω_r), the frequency where the derivative goes to zero is ω_r. The transmitted power at resonance P_r is then found from the $P(\omega)$ data at ω_r.

(3) Calculate $\Delta(\omega, \omega_r)$ and then determine τ vs Δ for the ensemble of ρ vs ω data.

(4) Perform a linear least-square fit to τ vs Δ. The slope of the fitted curve corresponds to Q_L.

As an example, this procedure was followed to obtain the loaded Q factor Q_L for an x-band TE011 transmission cavity using the same experimental technique outlined pre-

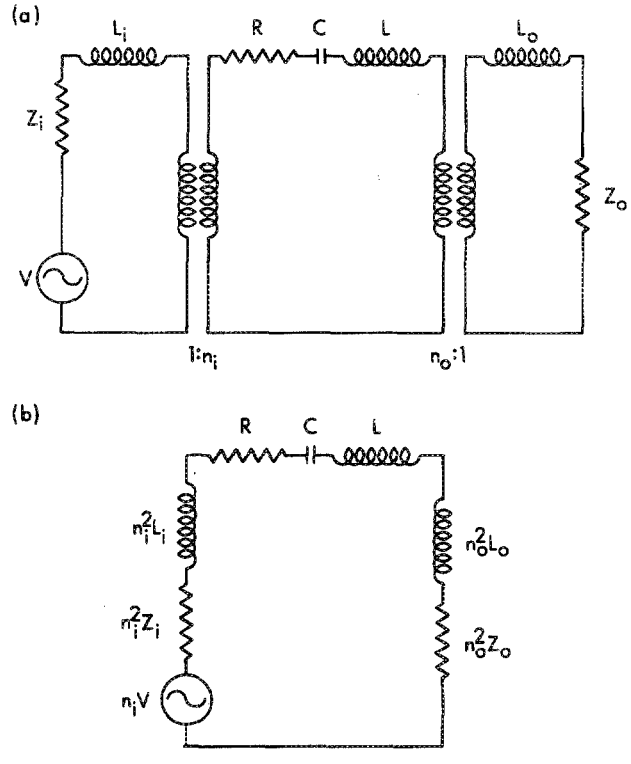

FIG. 3. (a) Equivalent circuit for an inductively coupled transmission cavity. (b) Transfer circuit for the circuit in (a). In both parts, V is the drive amplitude, Z_i and Z_0 are the input and output line impedances, L_i and L_0 are the iris inductances for the input and output irises, n_i and n_0 are the respective input and output iris coupling coefficients, and R, L, and C are the equivalent circuit cavity resistance, inductance, and capacitance, respectively.

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TABLE II. Representative data for a TE011 cylindrical transmission x-band cavity. Every 60th data point has been shown here. The nominal frequency is 9.967 GHz. The columns are for frequency f, transmitted power P, the reduced frequency parameter Δ, and the reduced cavity parameter τ.

f (GHz)	P (μW)	$\Delta(\times 10^{-4})$	τ
9.965 875	0.325	− 2.00	− 4.775
9.966 025	0.427	− 1.70	− 4.136
9.966 175	0.609	− 1.40	− 3.419
9.966 324	0.919	− 1.10	− 2.723
9.966 474	1.521	− 0.80	− 2.021
9.966 624	2.819	− 0.50	− 1.321
9.966 774	5.585	− 0.20	− 0.621
9.966 924	7.701	0.03	0.071
9.967 074	4.884	0.33	0.764
9.967 224	2.481	0.63	1.455
9.967 373	1.377	0.93	2.149
9.967 523	0.856	1.24	2.835
9.967 673	0.579	1.54	3.516
9.967 823	0.415	1.84	4.199

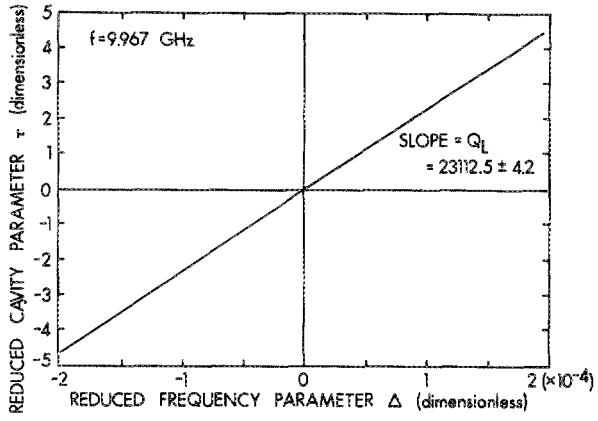

FIG. 4. Plot of the reduced cavity parameter τ vs the reduced frequency parameter Δ for the test transmission cavity. As in Fig. 2, the line reflects the parameter points, not the fitted curves. The slope of the fitted line corresponds to the cavity loaded Q, Q_L, and the error shown is the calculated standard deviation in Q_L from the least-square fit.

viously for the reflection cavity. In this case, however, it was the power transmitted through the cavity that was monitored by the scalar analyzer. Representative numerical results for the measured cavity frequency f (in GHz) and transmitted power P (in μW), and the reduced cavity parameters τ and Δ, are indicated in Table II for every 60th data point.

A plot of τ vs Δ for the transmission cavity is shown in Fig. 4. As before for the reflection cavity, only the actual parameter points are shown, not the fitted line. The slope of the fitted line corresponds to the cavity loaded Q, Q_L, and the indicated error is the calculated standard deviation of Q_L from the linear fit.

In this example, $Q_L = 23\,113$ and $\omega_r = 9.969$ GHz yield a frequency width at half-maximum of 0.43 MHz, roughly a fourth of the width of the frequency window used for data accumulation and analysis. A frequency window less than $\Delta\omega_{1/2}$ could have been used to get the same Q factor, and in fact any reasonable frequency window can be used to produce accurate results.

This technique is therefore quite useful in situations where the cavity resonance frequency ω_r or frequency half-width $\Delta\omega_{1/2}$ are not precisely known. In addition, accurate values for the Q error are quite helpful in determining systematic measurement errors which might go undetected with other measurement techniques.

ACKNOWLEDGMENTS

One of the authors (C.E.P.) is grateful to Dr. Z. Frait for introducing him to the reflection cavity Q measurement technique upon which much of this paper is based. This work was supported in part by the United States Office of Naval Research, Contract No. N00014-87-K-0325 P00001.

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