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Spin-Wave Instability Theory in Cubic Single Crystal Magnetic Insulators

I. General Theory[2])

By

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The spin-wave instability theory for magnetic insulators is extended to include the effect of magnetocrystalline anisotropy for any arbitrary microwave pumping configuration. The results are used to analyze previously published data for parallel pumping and subsidiary absorption. Part I presents the general theory. It is shown that previous effective field anisotropy formulations, which are valid for the analysis of ferromagnetic resonance, are not valid for spin-wave instability calculations if the microwave field has components transverse to the magnetizing field direction. General expressions are obtained for first-order process in cubic single crystals magnetized in an arbitrary direction in a (110) plane, with a general microwave pump polarization.

Es wird die Theorie der Spinwelleninstabilität für magnetische Isolatoren erweitert und schließt den Einfluß magnetokristalliner Anisotropie für jede beliebige Konfiguration des Mikrowellenpumpens ein. Die Ergebnisse werden benutzt, um früher publizierte Daten für paralleles Pumpen und zusätzliche Absorption zu analysieren. Teil I stellt die allgemeine Theorie dar. Es wird gezeigt, daß die bisherigen Formulierungen der Anisotropie des effektiven Feldes, die für die Analyse der ferromagnetischen Resonanz gültig sind, nicht mehr für Berechnungen der Spin-Wellen-Instabilität gültig sind, wenn das Mikrowellenfeld Komponenten senkrecht zur Richtung des Magnetisierungsfeldes besitzt. Allgemeine Ausdrücke werden für den Prozeß erster Ordnung in kubischen Einkristallen mit einer allgemeinen Mikrowellenpump-Polarisation erhalten, die in einer willkürlichen Richtung der (110)-Ebene magnetisiert sind.

1. Introduction

Following the discovery of ferromagnetic resonance (FMR) [1] and the derivation of the proper resonance condition by Kittel [2], the effect of magnetocrystalline anisotropy on the FMR was investigated by several authors. Kittel [2], Bickford [3], and Healey [4] included crystalline anisotropy in the resonance condition in terms of effective transverse demagnetizing factors. Artman [5] was later able to verify their results from a free-energy analysis of FMR. In several extensive papers, Haubenreisser and co-workers have applied the Kittel-Bickford-Healy (KBH) approach to first-order spin-wave instability processes in ferrites and obtained analytical results for the parallel pump and transverse pump threshold microwave fields, which included the effects of crystalline anisotropy [6 to 10].

In the present paper, the effect of cubic magnetocrystalline anisotropy is included in spin-wave instability theory, based on the Schlömann formalism [11] and a general effective field paproach similar to that used by Vittoria et al. [12], and Yakovlev and Burdin [13]. It is shown that the KBH effective transverse demagnetizing factors,

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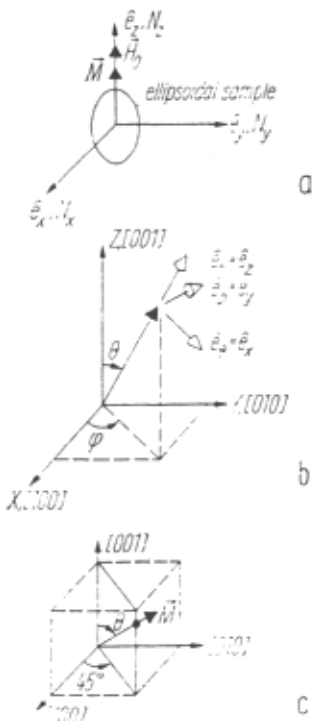
while adequate to explain FMR, are not adequate for a valid spin-wave instability analysis. This leads to significant modifications of the Haubenreisser theory in the case of transverse pumping.

The paper is divided into five sections. Section 2 evaluates the effective fields which are needed in the analysis. Section 3 incorporates these fields into the torque equation to obtain a classical equation of motion for the spin-wave amplitude for spin waves of non-zero wave vector $\boldsymbol{k}$. Section 4 considers the linear terms in the equation to obtain the spin-wave dispersion relation for materials with cubic anisotropy. Section 5 extends the analysis to include nonlinear terms and obtains a general expression for the instability threshold. Part II of this series considers special cases of parallel and transverse pumping for first-order processes. Results from the theory are compared, in detail, with available experimental data.

2. Effective Field Formulation

Consider an ellipsoidal-shaped, single crystal insulator in a configuration as depicted in Fig. 1. Fig. 1a shows the sample x, y, z coordinate system. The sample is assumed to be magnetized to saturation along the z-directed principal axis by an external magnetic field $\boldsymbol{H}_0$. The magnetization vector is denoted by $\boldsymbol{M}$. The ellipsoid is characterized by demagnetizing factors N_x, N_y, and N_z for the three principal axis directions x, y, and z, which are designated by unit vectors $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, and $\boldsymbol{e}_z$. Fig. 1b depicts the sample orientation relative to the cubic crystallographic X, Y, Z coordinate system. The orientation of the z-axis of the sample, and hence the magnetizing field $\boldsymbol{H}_0$ and $\boldsymbol{M}$, is described by the usual polar and azimuthal angles θ and φ. For convenience, the $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_\varphi$ unit vectors will be taken as coincident with $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$, respectively.

Most of the analysis in the sections to follow will be done for the case $\varphi = 45°$, so that $\boldsymbol{M}$ and $\boldsymbol{H}_0$ are constrained to lie in the $(1\bar{1}0)$ crystallographic plane. Under these conditions, depicted in Fig. 1c, $\theta = 0°$ corresponds to a [001] magnetized sample, $\sin^2\theta = \frac{2}{3}$ or $\theta \approx 54.74°$ corresponds to a [111] magnetized sample, and $\theta = 90°$ corresponds to a [110] magnetized sample. It is to be noted that the $\boldsymbol{e}_x$ unit vector is n the $[1\bar{1}0]$ direction.



It is assumed that $\boldsymbol{M}$ is composed of a static component parallel to $\boldsymbol{e}_z$ and a dynamic component $\boldsymbol{m}$, with $|\boldsymbol{m}| \ll M$. We may also write $\boldsymbol{M}$ in terms of a unit vector $\boldsymbol{\alpha}$,

$$\boldsymbol{M} = M\boldsymbol{\alpha} = M(\alpha_x \boldsymbol{e}_x + \alpha_y \boldsymbol{e}_y + \alpha_z \boldsymbol{e}_z)\,, \tag{1}$$

and expand $\boldsymbol{\alpha}$ in a Fourier series,

$$\boldsymbol{\alpha} = \sum_k \boldsymbol{\alpha}_k \, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\,. \tag{2}$$

When the wave vector $\boldsymbol{k}$ appears as a subscript or as a summation index, the vector notation will be deleted. The $\boldsymbol{\alpha}_k$ correspond to the classical spin-wave amplitudes.

The total effective field $\boldsymbol{H}_{\text{eff}}$ can be written as

$$\boldsymbol{H}_{\text{eff}} = \boldsymbol{H}_0 + \boldsymbol{h}_0 + \boldsymbol{H}_{\text{dip}} + \boldsymbol{H}_{\text{ex}} + \boldsymbol{H}_{\text{anis}}\,, \tag{3}$$

Fig. 1. Sample geometry and orientation, relative to the crystallographic cubic axes

where $\boldsymbol{H}_0 = H_0\boldsymbol{e}_z$ is the static external field and

$$\boldsymbol{h}_0 = h_0^x\boldsymbol{e}_x + h_0^y\boldsymbol{e}_y + h_0^z\boldsymbol{e}_z \tag{4}$$

is the applied external microwave field. The dipole field $\boldsymbol{H}_{\text{dip}}$ is given by

$$\boldsymbol{H}_{\text{dip}} = -4\pi M \sum_{k\neq 0} \frac{\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{\alpha}_k)}{k^2}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}} - 4\pi M\mathbf{N}\cdot\boldsymbol{\alpha}_0\,, \tag{5}$$

where $\mathbf{N}$ is the demagnetizing tensor in the x, y, z frame. Except for the $k = 0$ static demagnetizing field contribution, surface demagnetizing fields are neglected. Thus, the analysis is valid for k-values which are large compared to d^{-1}, where d is a nominal sample dimension. The exchange field $\boldsymbol{H}_{\text{ex}}$ is given by $D\nabla^2\boldsymbol{\alpha}$, where D is an exchange parameter on the order of 5.4×10^{-9} Oe cm^2 for YIG.

The effective anisotropy field $\boldsymbol{H}_{\text{anis}}$ is composed of a static component $\boldsymbol{H}_{\text{A}}$, which is essentially parallel to $\boldsymbol{M}$ and $\boldsymbol{H}_0$ under quasi-line-up conditions with $(\boldsymbol{M}\times\boldsymbol{H}_0)$ approximately zero and a dynamic component $\boldsymbol{h}_{\text{A}}$ which is related to $\boldsymbol{m}$. Both components can be derived from the expression for the magneto-crystalline anisotropy energy, F_{A}, which is to first order

$$F_{\text{A}} = K_1(\alpha_1^2\alpha_2^2 + \alpha_2^2\alpha_3^2 + \alpha_3^2\alpha_1^2)\,, \tag{6}$$

where K_1 is the usual first-order anisotropy constant and the $(\alpha_1, \alpha_2, \alpha_3)$ denote the direction cosines of $\boldsymbol{M}$ relative to the crystallographic axes.

The static component $\boldsymbol{H}_{\text{A}}$ is obtained from $\boldsymbol{H}_{\text{A}} = -\nabla_\alpha F_{\text{A}}/M$, where ∇_α denotes the gradient with respect to the direction cosines. For the special case $\varphi = 45°$, with $\boldsymbol{M}$ restricted to the $(1\bar{1}0)$ plane, one obtains

$$|\boldsymbol{H}_A| \approx \boldsymbol{H}_A\cdot\boldsymbol{e}_z = -\frac{2K_1}{M}\left(1 - \cos^4\theta - \frac{1}{2}\sin^4\theta\right). \tag{7}$$

Strictly speaking, $\boldsymbol{H}_{\text{A}}$ also has a small x-component, but this is cancelled by the effective field which results from the small deviations of $\boldsymbol{M}$ from the direction of $\boldsymbol{H}_0$. Under quasi-line-up conditions, the x-component may be neglected. The y-component in this special case of $\varphi = 45°$ is zero by symmetry. The dynamic, effective anisotropy field $\boldsymbol{h}_{\text{A}}$ is obtained by evaluating the variation of $\boldsymbol{H}_{\text{A}}$ with respect to the direction cosines α_1, α_2, and α_3. For $\varphi = 45°$, one obtains

$$\boldsymbol{h}_{\text{A}} = \begin{pmatrix} A_{xx} & 0 & A_{xz} \\ 0 & A_{yy} & 0 \\ A_{zx} & 0 & A_{zz} \end{pmatrix}\begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix} \tag{8}$$

with

$$A_{xx} = -\frac{2K_1}{M^2}(1 - 3\sin^2\theta)\left(1 - \frac{3}{2}\sin^2\theta\right), \tag{9}$$

$$A_{yy} = -\frac{2K_1}{M^2}\left(1 - \frac{3}{2}\sin^2\theta\right), \tag{10}$$

$$A_{xz} = A_{zx} = -\frac{2K_1}{M^2}\,3\sin\theta\cos\theta\left(1 - \frac{3}{2}\sin^2\theta\right), \tag{11}$$

$$A_{zz} = -\frac{2K_1}{M^2}\,3\sin^2\theta\left(2 - \frac{3}{2}\sin^2\theta\right). \tag{12}$$

It is to be noted that $\boldsymbol{h}_{\text{A}}$ contains x-, y-, and z-components. In the KBH approach, only x- and y-components appear. Furthermore, the dynamic part of the effective

anisotropy field has components which derive from the A_{zx} and A_{zz} elements of (8). These terms are absent in the KBH formulation. It will be evident from the analysis to follow that the A_{zx} factor is essential for a valid general instability theory.

It is easily seen, however, that these considerations are not important in evaluating the FMR frequency. Based on the above effective fields for the uniform mode with $k = 0$, one obtains a resonance condition,

$$\frac{\omega_0}{\gamma} = (H_x^{\text{eff}} H_y^{\text{eff}})^{1/2} , \tag{13}$$

with

$$H_x^{\text{eff}} = H_0 + (N_x - N_z)\, 4\pi M + (H_A - M A_{xx}) , \tag{14}$$

and

$$H_y^{\text{eff}} = H_0 + (N_y - N_z)\, 4M + (H_A - M A_{yy}) , \tag{15}$$

where γ denotes the gyromagnetic ratio. The expressions are identical to those from the KBH approach[4]) with $N_{x,y}$ replaced by $N_{x,y} + N_{x,y}^{\text{eff}}$,

$$4\pi M N_x^{\text{eff}} = H_A - M A_{xx} , \tag{16}$$

$$4\pi M N_y^{\text{eff}} = H_A - M A_{yy} . \tag{17}$$

The present analysis shows, however, that each of these effective transverse demagnetizing factors really derives from a static z-directed anisotropy field component and a dynamic transverse component. The distinction has no effect on the analytical result for the FMR frequency. As will be evident from the analysis to follow, the distinction is crucial to a valid spin-wave instability theory.

Two rather different kinds of second-order effects have not been included in the above effective field formulation of the crystalline anisotropy. The first concerns the second-order contribution to the anisotropy energy $K_2 \alpha_1^2 \alpha_2^2 \alpha_3^2$. The contribution of such terms to the effective fields is small and is neglected in the analysis presented here. The second concerns the higher-order contributions to the dynamic anisotropy field $\boldsymbol{h}_A$. The result in (8) to (12) concerns only the linear variation of $\boldsymbol{H}_A$. Haubenreisser and Linzen have included second-order variations in their analysis of transverse pumping [8]. Such terms, while not important for linearized FMR theory, may be important for nonlinear calculations. Such considerations will be discussed further in later sections.

3. Spin-Wave Equation of Motion

The approach and notation of this and subsequent sections closely follow the form as used by Schlömann [11] in his formulation of subsidiary absorption and parallel pumping theory, and by Patton [14] in his extension of the theory to include an arbitrary polarization of the microwave field. One writes the equation of motion for the unit vector $\boldsymbol{\alpha}$,

$$\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}t} = -\gamma \boldsymbol{\alpha} \times \boldsymbol{H}_{\text{eff}} , \tag{18}$$

and evaluates the equation of motion for the complex α_k defined by

$$\alpha_k = \alpha_k^x + i\alpha_k^y , \tag{19}$$

with

$$k_T = k_x + i k_y , \tag{20}$$

[4]) The correct expressions are given in equations (6) and (7) of [3].

$$H = H_0 - 4\pi M N_z + H_A \,, \tag{21}$$

$$h_{x,y} = h_0^{x,y} - 4\pi M N_{x,y}\alpha_0^{x,y} + M A_{xx,yy}\alpha_0^{x,y} \,, \tag{22}$$

$$h = h_x + i h_y \,. \tag{23}$$

One obtains an expression for $\dot{\alpha}_k$ of the form

$$\dot{\alpha} = i\gamma(Q_0 + Q_A) \,, \tag{24}$$

where Q_0 is identical with the expression given in equation (5) of [14],

$$Q_0 = [H + h_0^z + 4\pi M N_z(1 - \alpha_0^z)]\,\alpha_k - \alpha_k^z h + \\ + D\sum_{k'}(k')^2(\alpha_{k'}\alpha_{k-k'}^z - \alpha_{k'}^z\alpha_{k-k'}) + 4\pi M\sum_{k'\neq 0}\boldsymbol{k}'\cdot\boldsymbol{\alpha}_{k'}\frac{k_T'\alpha_{k-k'}^z - k_z'\alpha_{k-k'}}{(k')^2} \,. \tag{25}$$

Here,

$$a_k^z = \delta_{k0} - \tfrac{1}{2}\sum_{k'}\alpha_{k'}\alpha_{k'-k}^* \,. \tag{26}$$

The Q_A term contains all the explicit anisotropy effects (note that anisotropy is also contained implicitly in the expressions for H and $h_{x,y}$). Q_A is given by

$$Q_A = -\frac{MA_+}{2}\sum_{k''\neq 0}\alpha_{k-k''}^z\alpha_{k''} - \frac{MA_-}{2}\sum_{k''=0}\alpha_{k-k''}^z\alpha_{-k''}^* + \\ + MA_{zz}\left[(\alpha_0^z - 1)\,\alpha_k + \sum_{k''\neq 0}\alpha_{k-k''}\alpha_{k''}^z\right] + \\ + MA_{xz}\left[\sum_{k''\neq 0}\alpha_{k-k''}\alpha_{k''}^x - \sum_{k''\neq 0}\alpha_{k-k''}^z\alpha_{k''}^z - (\alpha_0^z - 1)\,\alpha_k^z\right], \tag{27}$$

where $A_\pm = A_{xx} \pm A_{yy}$.

The next step is to evaluate each of the sum terms in Q_0 and Q_A, and retain only those terms which are linear in the α_k with $k \neq 0$. There are two classes of such terms, those which contain no α_0 factors and those which do contain such factors. The result is

$$\dot{\alpha}_k = i[(A_k^0 + A_k^A + C_k^0 + C_k^A)\,\alpha_k + \\ + (B_k^0\,e^{2i\varphi_k} + B_k^A + D_k^0 + D_k^A)\,\alpha_{-k}^* + \gamma h_0^z\alpha_k] \,. \tag{28}$$

The "0" superscripts denote terms which derive from Q_0; the "A" superscripts denote terms which derive from Q_A. The A_k and B_k denote terms linear in the α_k with $k \neq 0$ and no α_0 factors; the C_k and D_k denote the terms which do contain α_0 factors. Recall that α_0 is the complex uniform precession mode amplitude, which can be large under conditions of transverse pumping (i.e. h_0^x or h_0^y large). It is important, therefore, to retain such terms in the analysis. The various coefficients are

$$A_k^0 = \gamma H + \gamma D k^2 + \tfrac{1}{2}\omega_m \sin^2\theta_k \,, \tag{29}$$

$$B_k^0 = \tfrac{1}{2}\omega_m \sin^2\theta_k \,, \tag{30}$$

$$C = -\tfrac{1}{2}\omega_m \cos\theta_k \sin\theta_k\,[e^{-i\varphi_k}\alpha_0 + e^{i\varphi_k}\alpha_0^*] + \\ + \tfrac{1}{4}\omega_m[2N_z + (3\cos^2\theta_k - 1)]\,|\alpha_0|^2 + \tfrac{1}{2}\gamma h'\alpha_0^* \,, \tag{31}$$

$$D_k^0 = \tfrac{1}{2}\gamma D k^2\alpha_0^2 - \omega_m \cos\theta_k \sin\theta_k\,e^{i\varphi_k}\alpha_0 + \\ + \tfrac{1}{4}\omega_m(2\cos^2\theta_k\,\alpha_0^2 - \sin^2\theta_k\,e^{2i\varphi_k}\,|\alpha_0|^2) + \tfrac{1}{2}\gamma h'\alpha_0 \,, \tag{32}$$

$$A_k^{\mathrm{A}} = -\frac{\gamma M A_+}{2}, \tag{33}$$

$$B_k^{\mathrm{A}} = -\frac{M A_-}{2}, \tag{34}$$

$$\frac{C_k^{\mathrm{A}}}{\gamma} = M A_{zx}\left[\alpha_0 + \alpha_0^* - \frac{1}{2}\alpha_0(\alpha_0^*)^2\right] - \frac{1}{2} M A_{zz} |\alpha_0|^2 + \\ + \frac{1}{2} M A_+ |\alpha_0|^2 + \frac{1}{4} M A_-(\alpha_0^*)^2, \tag{35}$$

and

$$\frac{D_k^{\mathrm{A}}}{\gamma} = M A_{zx}\left(\alpha_0 - \frac{1}{2}\alpha_0^2\alpha_0^*\right) - \frac{1}{2} M A_{zz}\alpha_0^2 + \frac{1}{4} M A_+\alpha_0^2 + \frac{1}{4} M A_- |\alpha_0|^2 . \tag{36}$$

In the above, θ_k and φ_k refer to the direction of the wave vector $\boldsymbol{k}$, in terms of the usual spherical coordinate system relative to the x, y, z reference frame. The complex field h' is given by

$$h' = h_x' + i h_y', \tag{37}$$

$$h_{x,y}' = h_0^{x,y} - 4\pi M N_{x,y}\alpha_0^{x,y} . \tag{38}$$

The anisotropy terms included in h, as specified by (22) and (23), have been considered explicitly. The quantity ω_{m} is defined by $\omega_{\mathrm{m}} = \gamma 4\pi M$.

Note that in the limit of no external microwave field, h' and α_0 are both zero. In this limit, one obtains

$$\dot{\alpha}_k = i[(A_k^0 + A_k^{\mathrm{A}})\,\alpha_k + (B_k^0 e^{2i\theta_k} + B_k^{\mathrm{A}})\,\alpha_{-k}^*] . \tag{39}$$

Diagonalization of this equation by the Holstein-Primakoff [15] (see also [16]) procedure leads to the spin-wave dispersion. If only a z-directed microwave field is applied, an additional $i\gamma h_0^z \alpha_k$ term appears on the right-hand side of (39). This case is parallel pumping under the additional influence of crystalline anisotropy. If transverse microwave field components are present, h' and α are nonzero. These special cases will be considered in Part II of this series.

4. Spin-Wave Dispersion Relation

If only the A_k and B_k terms in the equation of motion are considered, one obtains

$$\dot{\alpha}_k = i(A_k^{\mathrm{a}}\alpha_k + B_k^{\mathrm{a}} e^{2i\varphi_a}\alpha_{-k}^*) , \tag{40}$$

with

$$A_k^{\mathrm{a}} = A_k^0 - K_+ , \tag{41}$$

$$B_k^{\mathrm{a}} = \sqrt{(B_k^0 \cos 2\varphi_k - K_-)^2 + (B_k^0 \sin 2\varphi_k)^2} , \tag{42}$$

$$\varphi_a = \frac{1}{2}\tan^{-1}\left[\frac{B_k^0 \sin 2\varphi_k}{B_k^0 \cos 2\varphi_k - K_-}\right], \tag{43}$$

and

$$K_\pm = \frac{\gamma M A_\pm}{2} . \tag{44}$$

This equation of motion can then be diagonalized by a Holstein-Primakoff type transformation [15, 16], to obtain a spin-wave frequency ω_k specified by

$$\omega_k^2 = (A_k^0 + B_k^0 - \gamma M A_{xx})(A_k^0 - B_k^0 - \gamma M A_{yy}) - 2\gamma M A_- B_k^0 \sin^2\varphi_k . \tag{45}$$

Recall that A_k^0 contains the static anisotropy field H_A incorporated in the γH term. The quantities $(H_A - MA_{xx})$, $(H_A - MA_{yy})$, and $M(A_{xx} - A_{yy})$ which occur in ω_k are therefore identical to those in the dispersion result by Haubenreisser and Linzen [7] based on the Kittel-Bickford-Healey approach.

If the anisotropy is set to zero, one obtains the usual Holstein-Primakoff dispersion relation for an isotropic ferromagnetic insulator which depends only on the polar spin-wave angle θ_k. The main effect of anisotropy, apart from small frequency shifts due to the modified effective fields, is to introduce a dependence of ω_k on the azimuthal spin-wave angle φ_k. This derives from the $\sin^2 2\varphi_k$ term in (45). It is illustrative to examine the dependence of ω_k on θ_k and φ_k in more detail. This will have a direct bearing on the available spin-wave states when instability processes are considered. If the θ_k and φ_k dependences are included explicitly in ω_k, one obtains

$$\frac{\omega_k^2}{\gamma^2} = (H + Dk^2 + 4\pi Ms \sin^2\theta_k - MA_{xx})(H + Dk^2 - MA_{yy}) - \\ - 4\pi MMA_- \sin^2\theta_k \sin^2\varphi_k \,. \tag{46}$$

Thus, it is seen that the $\theta_k = 0$ branch of the spin-wave manifold is not a function of φ_k. Branches for other values $0 < \theta_k \leqq \pi/2$ will depend on φ_k. Increasing φ_k from 0 to $\pi/2$ can either raise or lower the spin-wave frequency, depending on the sign and magnitude of A_-.

Recall that the specific crystallographic configuration which is considered here is for the magnetization and external static field in the (1$\bar{1}$0) plane, at an angle θ (not to be confused with θ_k) to the [001] direction. For the special case of a negative first-order anisotropy constant $K_1 < 0$, which applies to yttrium iron garnet and a number of other important micro-wave ferrites, MA_- is given by

$$MA_- = -6H_k \sin^2\theta(1 - \tfrac{3}{2}\sin^2\theta) \,, \tag{47}$$

where H_k is a positive anisotropy field defined as $|K_1|/M$. For yttrium–iron garnet, H_k is approximately 43 Oe. In Fig. 2, the quantity MA_-/H_k is plotted as a function of the magnetization orientation angle θ. Note that the φ_k dependence of ω_k vanishes for $\theta = 0$ and $\theta = 54.74°$, which correspond to [001] and [111] orientations of the magnetization, respectively. The φ_k dependence is maximum for the [110] orientation, with $\theta = 90°$. In this case, it is seen that ω_k must decrease as φ_k increases from zero to $\pi/2$. The spin-wave manifold is schematically illustrated in Fig. 3, in the form of a perspective sketch. For $\theta < 54.74°$, the φ_k dependence would be reversed, i.e. ω_k would increase with φ_k. The detailed behaviour of the manifold and available states for instability as a function of applied field will be examined more closely in Part II of this series.

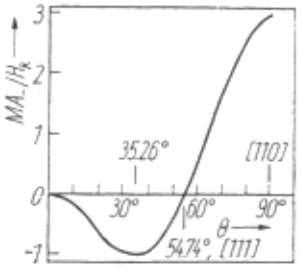


Fig. 2

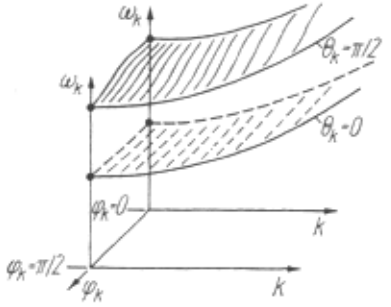


Fig. 3

Fig. 2. Orientation dependence of the φ_k term in the spin-wave dispersion relation for ω_k, where θ is the angle between the (110) plane magnetization vector and the [001] direction

Fig. 3. Schematic representation of the dispersion relation for [110] magnetized materials with $K_1 < 0$

5. Evaluation of Instability Processes

The same transformations that were used to diagonalize the linear terms in the equation of motion to obtain $\dot{b}_k = i\omega_k b_k$ may be applied to the C_k and D_k terms. One obtains

$$\dot{b}_k = i[\omega_k b_k + (F_k^a + F_k^{az})\, b_k + (G_k^a + G_k^{az})\, b_{-k}^*] \,. \tag{48}$$

The F_k terms lead only to a small modulation and shift in the spin-wave frequency and can be neglected. The G_k terms lead to the unstable growth of the b_k spin-wave amplitude for sufficiently high power levels, as will be discussed shortly. The G_k terms are given by

$$G_k^a = \frac{e^{2i\varphi_a}}{2\omega_k} \{D_k^a (A_k^a + \omega_k)\, e^{-2i\varphi_a} + (D_k^a)^* (A_k^a - \omega_k)\, e^{2i\varphi_a} - [C_k^a + (C_k^a)^*]\, B_k^a\} \,, \tag{49}$$

$$G_k^{az} = \frac{e^{2i\varphi_a}}{2\omega_k} B_k^a \gamma h_0^z \,, \tag{50}$$

with $C_k^a = C_k^0 + C_k^A$ and $D_k^a = D_k^0 + D_k^A$. These expressions have precisely the same form as the coefficients in the transformed $\dot{b}_k$ equation of motion of [14], with appropriate redefinitions of the A_k, B_k, C_k, D_k, etc., to include anisotropy.

All that remains, in terms of the general formulation, is to relate the spin-wave instability threshold to the G_k coefficient. This is most easily demonstrated by considering a b_k equation of motion in the form

$$\dot{b}_k = i(\Omega_k b_k + G_k b_{-k}^*) \,, \tag{51}$$

along with the complimentary equation for $\dot{b}_{-k}^*$. A complex spin-wave frequency, $\Omega_k = \omega_k + i\eta_k$, has been introduced, where η_k denotes the spin-wave relaxation rate for the particular k-value under consideration. Note that $G_k = G_{-k}$ is satisfied. Following the procedure in [11, 14], the equation of motion is solved by noting that G_k is of the form

$$G_k = \sum_{n=-2}^{+2} G_k^{(n)}\, e^{i\omega n t} \,. \tag{52}$$

Solutions are obtained in terms of a transformed variable $b_{k0}^{(n)}(t)$ defined by

$$b_k(t) = \sum_{n=-2}^{+2} b_{k0}^{(n)}(t)\, e^{i\omega n t/2} \,. \tag{53}$$

For each $b_{k0}^{(n)}$, one obtains a second-order differential equation,

$$\frac{d^2}{dt^2} b_{k0}^{(n)} + 2\eta_k \frac{d}{dt} b_{k0}^{(n)} + \left[\left(\omega_k - \frac{\omega n}{2}\right)^2 + \eta_k^2 - |G_k^{(n)}|^2\right] b_{k0}^{(n)} = 0 \,. \tag{54}$$

This equation has a solution,

$$b_{k0}^{(n)}(t) = \text{const}\; e^{\varkappa_n t} \,, \tag{55}$$

where $\varkappa_n$ is given by

$$\varkappa_n = \sqrt{|G_k^{(n)}|^2 - \left(\omega_k - \frac{\omega n}{2}\right)^2} - \eta_k \,. \tag{56}$$

Since $|G_k^{(n)}|^2$ will increase with the applied external microwave field, the condition

$\varkappa_n = 0$ will determine the threshold condition for the unstable growth of spin-waves. The lowest threshold will be for $\omega_k \approx \omega n/2$ and $|G_k^{(n)}| = \eta_k$.

One new feature of the above differential equation solution is that it applies to all of the harmonic terms in (52) and (53) for arbitrary n. In the previous analyses, the solution was only obtained for first-order processes with $n = 1$. Of course, under usual experimental conditions for parallel pumping, subsidiary absorption, and oblique pumping, the lowest threshold is for first-order processes with $n = 1$ and ω_k at one half the pump frequency. This is the most reasonable situation to study in detail.

The different harmonic terms in G_k^a are identified by realizing that the time dependent terms occur only in the C_k^a and the D_k^a expressions. These are contained in the various combinations of α_0 and α_0^* which occur in C_k^a and D_k^a. Recall that α_0 is the complex uniform precession amplitude. It may be written as

$$\alpha_0 = \alpha_{LO}\, e^{i\omega t} + \alpha_{AO}\, e^{-i\omega t}\,, \tag{57}$$

where α_{LO} and α_{AO} are complex time independent coefficients which characterize the Larmor and anti-Larmor parts of the uniform precession, respectively. These amplitudes are, of course, driven by the transverse components of the applied external microwave field. Let this field be specified by

$$h_0^{x,y} = h_0 a_{x,y} \cos(\omega t + \delta_{x,y})\,, \tag{58}$$

$$h_0 = h_0 a_z \cos(\omega t)\,. \tag{59}$$

Various combinations of a_x, a_y, a_z, δ_x, and δ_y can be used to describe most experimentally attainable drive configurations. Note that the parallel pumping contribution to G_k is due to the G_k^{az} term given by (50), which has only a contribution for $n = 1$. The $n = 1$ contributions to G_k^a come from a tedious analysis of the harmonics contained in C_k^0, C_k^A, D_k^0, D_k^A, and their complex conjugates. The final result for $G_k^{(1)}$ is given by

$$\begin{aligned} G_k^{(1)} = -\frac{e^{2i\varphi_a}}{2\omega_k} \{ & [\omega_m \sin\theta_k \cos\theta_k e^{i\varphi_k} + \omega_{zx}][A_k^a + \omega_k]\, e^{-2i\varphi_a}\alpha_{LO} + \\ & + [\omega_m \sin\theta_k \cos\theta_k\, e^{-i\varphi_k} + \omega_{zx}]\,[A_k^a - \omega_k]\, e^{2i\varphi_a}\alpha_{AO} - \\ & - [\omega_m \sin\theta_k \cos\theta_k\, (e^{-i\varphi_k}\alpha_{LO} + e^{i\varphi_k}\alpha_{AO}^*) - 2\omega_{zx}\alpha_{LO} - 2\omega_{zx}\alpha_{AO}^*]\, B_k^a + \\ & + B_k^a h_0 a_z\}\,, \end{aligned} \tag{60}$$

where MA_{zx} has been expressed as a frequency, $\omega_{zx} = \gamma MA_{zx}$.

There are several features of this result that deserve emphasis. First of all, the expression reduces to that obtained in [14] for isotropic materials if the anisotropy is set to zero. In this limit, one has $A_k^a = A_k^0$, $B_k^a = B_k^0$, $\varphi_k = \varphi_a$, and $\omega_{zx} = 0$. Second, it is important to note that the effects of anisotropy on the coupling terms for transverse pumping derive from the ω_{zx} or the A_{zx} terms in the dynamic effective anisotropy field of Section 2. These terms are completely ignored in the theory of Haubenreisser and Linzen [8] for transverse pumping. This is presumably the motivation in that theory to consider second-order contributions to the dynamic effective anisotropy field. Third, in the case of pure parallel pumping only the last term on the right-hand side of (60) survives, and one obtains results identical to those of Haubenreisser and Linzen [7] for parallel pumping. Parallel pumping and transverse pumping will be considered in Part II.

It is useful to redefine several quantities in order to obtain more compact expressions for later numerical evaluation. Express the Larmor and anti-Larmor amplitudes as

$$\alpha_{LO} = \frac{\gamma h_0}{2}\, q_L\,, \tag{61}$$

$$\alpha_{AO} = \frac{\gamma h_0}{2} q_A \,, \tag{62}$$

respectively. The spin-wave relaxation rate η_k is related to the spin-wave linewidth ΔH_k by

$$\Delta H_k = \frac{2\eta_k}{\gamma}. \tag{63}$$

If we define a quantity W according to

$$G_k^{(1)} = -\frac{\omega_m}{\omega}\frac{\gamma h_0}{2} e^{2i\varphi_k} W \,, \tag{64}$$

the condition $|G_k^{(1)}| = \eta_k$ yields a rather simple working equation for the instability threshold of a given spin-wave $\boldsymbol{k}$ at $\omega_k = \omega/2$,

$$h_c = \frac{\omega}{\omega_m}\frac{\Delta H_k}{|W|}. \tag{65}$$

Note that h_c in (65) is a function of the wave vector $\boldsymbol{k}$. This comes about because both ΔH_k and $|W|$ are functions of $\boldsymbol{k}$ [17 to 20]. Subject to the constraints on the available spin-wave states at $\omega_k = \omega/2$ for any particular static field, sample field, sample shape, and sample orientation, the observed threshold h_{crit} will be the h_c for the particular (k, θ_k, φ_k) combination at $\omega_k = \omega/2$ which minimizes $\Delta H_k/|W|$. As will be evident from Part II, the dependence of ΔH_k on $\boldsymbol{k}$, the modifications in the dispersion due to anisotropy, and the effect of anisotropy on $G_k^{(1)}$ all have profound, interrelated effects on the observed h_{crit}.

References

[1] J. H. E. GRIFFITHS, Nature (London) **158**, 670 (1946).
[2] C. KITTEL, Phys. Rev. **73**, 155 (1948).
[3] L. R. BICKFORD, Phys. Rev. **78**, 449 (1950).
[4] D. W. HEALEY, Phys. Rev. **86**, 1009 (1952).
[5] J. O. ARTMAN, Phys. Rev. **105**, 62 (1957).
[6] W. HAUBENREISSER, D. LINZEN, and E. GLAUCHE, phys. stat. sol. **1**, 764 (1961).
[7] W. HAUBENREISSER and D. LINZEN, phys. stat. sol. **2**, 734 (1962).
[8] W. HAUBENREISSER and D. LINZEN, phys. stat. sol. **3**, 133 (1963).
[9] W. HAUBENREISSER, TH. KLUPSCH, and F. VOIGT, phys. stat. sol. (b) **66**, 9 (1974).
[10] W. HAUBENREISSER, TH. KLUPSCH, and F. VOIGT, phys. stat. sol. (b) **73**, 11 (1976).
[11] E. SCHLÖMANN, Raytheon Tech. Rep. R-48, Waltham (Ma.) 1959.
[12] C. VITTORIA, G. C. BAILEY, R. C. BARKER, and A. YELON, Phys. Rev. B **7**, 2112 (1973).
[13] Y. M. YAKOVLEV and Y. N. BURDIN, Soviet Phys. — Solid State **16**, 301 (1974).
[14] C. E. PATTON, J. appl. Phys. **40**, 2837 (1969).
[15] T. HOLSTEIN and H. PRIMAKOFF, Phys. Rev. **58**, 1098 (1940).
[16] M. SPARKS, Ferromagnetic Relaxation Theory, McGraw Hill Publ. Co., New York 1964.
[17] C. E. PATTON, J. appl. Phys. **41**, 431 (1970).
[18] Y. M. YAKOVLEV, Y. N. BURDIN, Y. R. SHILNIKOV, and T. N. BUSHUEVA, Soviet Phys. — Solid State **12**, 2475 (1971).
[19] C. E. PATTON and W. JANTZ, IEEE Trans. Magnetics **14**, 698 (1978).
[20] J. C. SETHARES, M. H. SEAVEY, T. G. PURNHAGEN, and M. R. STIGLITZ, J. appl. Phys. **35**, 898 (1964).

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